Principles of Fraction Addition and Subtraction

Elementary algebra is essentially basic arithmetic with numerals replaced with algebraic variables. The student that fully understands the number principles involved in operations with fractions will learn algebra much easier than the student who does not. The principles of this section will prepare the student for subsequent studies in algebra.

Like Fractions

The term **like fractions** is used in this text to describe two or more fractions that have the same denominator.

The fractions $\frac{3}{8}$, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{5}{8}$, and $\frac{13}{8}$ are like fractions since they have the same denominator.

In all of the like fractions above, the denominator is 8. Each of these fractions represents pieces of one or more whole units that have been divided into 8 equal-size pieces.

**Example**

How would you represent $\frac{3}{8} + \frac{2}{8}$ with pieces of whole units?

Each whole unit would have to be divided into 8 pieces of equal size. 3 pieces from the first unit would be added to 2 pieces from the second whole unit. The result would be 5 pieces or $\frac{5}{8}$. This is illustrated here.

In Example 1, the two like fractions could be combined because they had the same denominator. In general, only like fractions can be combined using addition.

**MATH FACT**

- Only fractions with the *same* denominators can be combined using addition or subtraction.
**Example**  Can the fractions, $\frac{3}{8}$ and $\frac{1}{4}$, be added in the form they are in?

Since these fractions are unlike, they cannot be added. The fraction $\frac{1}{4}$ must be built into an equivalent fraction with a denominator of 8.

**Procedure to Add or Subtract Fractions**

When fractions that are added or subtracted do not have the same denominators, they must be changed in form using the “building” process discussed in Section 2.2. In Example 2, $\frac{3}{8}$ and $\frac{1}{4}$ must be added in the following way:

$$\frac{3}{8} + \frac{1}{4} = \frac{3}{8} + \frac{1 \times 2}{4 \times 2} = \frac{3}{8} + \frac{2}{8} = \frac{3 + 2}{8} = \frac{5}{8}$$

The procedure for adding and subtracting fractions is restated here.

<table>
<thead>
<tr>
<th>PROCEDURE TO ADD OR SUBTRACT FRACTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1.</strong> If the fractions have different denominators, find a common denominator and build new equivalent fractions with the common denominator.</td>
</tr>
<tr>
<td><strong>2.</strong> Combine the numerators of all of these like fractions with the addition or subtraction operations present. Write this result over the common denominator.</td>
</tr>
<tr>
<td><strong>3.</strong> If possible, reduce the fraction to lowest terms.</td>
</tr>
</tbody>
</table>
Example

Add \( \frac{3}{15} + \frac{7}{15} \).

Since these are like fractions, they may be added in the form that they are in.

\[
\frac{3}{15} + \frac{7}{15} = \frac{3 + 7}{15} = \frac{10}{15}
\]

\( \frac{10}{15} \) reduces to \( \frac{2}{3} \).

Example

Add \( \frac{3}{5} + \frac{6}{15} \).

Since these fractions are unlike, we must build \( \frac{3}{5} \) into an equivalent fraction with a denominator of 15. Since \( 5 \times 3 \) is 15, we multiply the numerator and the denominator of \( \frac{3}{5} \) by 3.

\[
\frac{3}{5} + \frac{6}{15} = \frac{3 \times 3}{5 \times 3} + \frac{6}{15}
\]

\[
= \frac{9}{15} + \frac{6}{15}
\]

\[
= \frac{9 + 6}{15} = \frac{15}{15} = 1
\]

Example

Subtract \( \frac{4}{15} - \frac{5}{30} \).

We must build the fraction \( \frac{4}{15} \) into an equivalent fraction with a denominator of 30. Multiply the numerator and denominator by 2.

\[
\frac{4}{15} - \frac{5}{30} = \frac{4 \times 2}{15 \times 2} - \frac{5}{30}
\]

\[
= \frac{8}{30} - \frac{5}{30}
\]

\[
= \frac{8 - 5}{30} = \frac{3}{30}
\]

\( \frac{3}{30} \) reduces to \( \frac{1}{10} \).
In the previous examples, it was easy to determine what the common denominator was. Many fraction addition and subtraction problems are not as simple, especially algebraic fraction problems. Thus, we need to learn the principles of fraction addition and subtraction that are on the following pages.

**Factors and the Least Common Multiple (LCM)**

A *factor* of a given whole number is another whole number which divides in evenly with no remainder. Finding factors of denominators is especially important in adding and subtracting algebraic fractions.

**Example**

What are the factors of 32?

The factors of 32 are 1, 32, 2, 16, 4, and 8.

In Example 6, the factors were found by dividing 32 by every possible whole number from 1 to 32, beginning with 1. If a whole number divided evenly into 32, then both the divisor *and* the quotient were factors. This procedure is summarized here.

<table>
<thead>
<tr>
<th>PROCEDURE FOR FINDING FACTORS OF A GIVEN WHOLE NUMBER</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The given whole number itself and 1 will always be factors.</td>
</tr>
<tr>
<td>2. Divide the whole number by every possible whole divisor, beginning with 2. If the divisor divides in evenly, then both the divisor and the quotient will be factors.</td>
</tr>
<tr>
<td>3. Stop dividing by possible factors when the divisor is greater than or equal to the quotient. A continuation beyond this point will only produce factors which have already been determined.</td>
</tr>
</tbody>
</table>
Example Find all the factors of 40.

1 and 40 will be factors.

\[
\begin{align*}
40 \div 2 &= 20 & \text{2 and 20 are factors.} \\
40 \div 3 &= 13 \text{ r } 1 & \text{3 is not a factor.} \\
40 \div 4 &= 10 & \text{4 and 10 are factors.} \\
40 \div 5 &= 8 & \text{5 and 8 are factors.} \\
40 \div 6 &= 6 \text{ r } 4 & \text{6 is not a factor. Stop at this point since the divisor 6} \\
& & \text{is greater than or equal to the quotient of 6.}
\end{align*}
\]

The factors of 40 are 1, 2, 4, 5, 8, 10, 20 and 40.

Example Find all the factors of 101.

1 and 101 will be factors.

101 is not evenly divisible by 2, 3, 4, 5, 6, 7, 8, 9 or 10

Since \(101 \div 10 = 10 \text{ r } 1\), it is not necessary to check any more counting numbers greater than 10. The only factors of 101 are 1 and 101.

Primes and Composites

In Example 8, the only factors consisted of 1 and 101 itself. When a whole number that is greater than 1 has no factors other than 1 and the whole number itself, the counting number is said to be prime.

<table>
<thead>
<tr>
<th>PRIME NUMBERS</th>
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</thead>
<tbody>
<tr>
<td>● A prime number is a whole number greater than 1 that has no factors other than the number itself and 1.</td>
</tr>
<tr>
<td>● No whole number other than 1 and the prime number itself divides in evenly into a prime number.</td>
</tr>
</tbody>
</table>

If a counting number is divisible by some counting number other than 1 and the number itself, then the number is called a composite number.
Example

Is 121 prime or composite?

Since $121 \div 11 = 11$, 121 is composite.

---

### HOW TO TEST A NUMBER TO SEE IF IT IS PRIME

1. Test all possible whole numbers for divisibility starting with 2.
2. If a whole number divides in evenly, the number being tested is composite. If not, then the number being tested is prime.

Testing a number to see if it is prime can be time consuming. The time spent checking for divisibility of all the possible factors can be shortened dramatically by using the following divisibility tests.

---

### DIVISIBILITY TESTS

- A number is divisible by 2 if it is even or ends in an even digit.
- A number is divisible by 3 if the sum of the digits of the number is divisible by 3.
- A number is divisible by 5 if it ends in 5 or 0.
- If a number is not divisible by 2, then it is not divisible by a multiple of 2.
- If a number is not divisible by 3, then it is not divisible by any multiple of 3.
- If a number is not divisible by 5, then it is not divisible by any multiple of 5.

---

Example

Is 30,021 divisible by 3?

Since the digits add up to $3 + 0 + 0 + 2 + 1 = 6$ and 6 is divisible by 3, we can conclude that 30,021 is divisible by 3.

Example

Is 344,099 divisible by 14?

Since 344,099 is not divisible by 2, it will not be divisible by any multiple of 2. 344,099 is not divisible by 14 because 14 is multiple of 2.
Example

Is 103 prime or composite?

Since 103 is odd, it is not divisible by 2. Also, it will not be divisible by any multiple of 2.

Because the digits add up to $1 + 0 + 3 = 4$, a number that is not divisible by 3, 103 will not be divisible by 3 nor any multiple of 3.

Since 103 does not end in 5 or 0, it is not divisible by 5 nor any multiple of 5.

Since 103 is not divisible by 2, 3, or 5 we can conclude that 103 is not divisible by any multiple of 2 or 3 or 5. Therefore, 103 is not divisible by 4, 6, 8, 9, or 10. We have to check for divisibility by 7 however.

$103 \div 7 = 4 \text{ r } 5$ Thus, 7 is not a factor.

Since $103 \div 10 = 10 \text{ r } 3$, and the divisor is greater than or equal to the quotient, we need not check any more factors larger than 10. Since no factors divided into 103, we can conclude that 103 is prime.

Prime Factorization

In order to add algebraic fractions, it is necessary to completely factor each denominator. Then, use the factored denominators to determine what the least common denominator is. For numerical fractions, the same procedure may be used. This process of completely factoring the denominators is known as finding the prime factorizations. Prime factorizations will be used to determine the least common denominator of several fractions.

Example

Find the prime factorization of 12.

$12 = 2 \times 6$ This is not a prime factorization because 6 is not prime.

We can, however, write $12 = 2 \times 6$ as $12 = 2 \times 2 \times 3$.
Note that 6 was replaced with $2 \times 3$.

The prime factorization of 12 is $12 = 2 \times 2 \times 3$.

We will use exponents to represent repeated factors and write this result as $12 = 2^2 \times 3$.

Note that in this example 12 has been rewritten as a product of prime numbers. The procedure for finding the prime factorization of any counting number is given here.
PROCEDURE FOR WRITING THE PRIME FACTORIZATION OF A NUMBER

1. Write the number as a product of any two or more factors.

2. Rewrite any factor that is not a prime number as a product of two or more factors.

3. Use exponent notation for any repeated factors.

Example

Write the prime factorization of 450.

\[ 450 = 9 \times 50 = 3 \times 3 \times 50 \]

\[ 450 = 3 \times 3 \times 50 = 3 \times 3 \times 2 \times 25 \]

\[ 450 = 3 \times 3 \times 2 \times 25 = 3 \times 3 \times 2 \times 5 \times 5 \]

The final result is written as \( 450 = 3^2 \times 2 \times 5^2 \).

Example

Find the prime factorization of 144.

First, write 144 as the product of any two or more factors.

\[ 144 = 2 \times 72 \]

The factor 2 is prime, but 72 is not. Thus, replace the factor 72 with the product of any two or more factors.

\[ 144 = 2 \times 72 = 2 \times 2 \times 36 \]

Continue to rewrite any factor that is not prime as the product of two or more factors until all factors are prime numbers. Write the result in exponent form.

\[ 144 = 2 \times 2 \times 36 = 2 \times 2 \times 2 \times 18 \]

\[ 144 = 2 \times 2 \times 2 \times 18 = 2 \times 2 \times 2 \times 2 \times 9 \]

\[ 144 = 2 \times 2 \times 2 \times 2 \times 9 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \]

The final result is written as \( 144 = 2^4 \times 3^2 \).
Least Common Multiple

The least common multiple of two or more given numbers consists of the smallest multiple that is common to all of the given numbers. Least common multiple is abbreviated LCM. The LCM will be used in fraction addition and subtraction.

Example  What is the least common multiple (LCM) of 2, 4 and 6.

The first twelve multiples of 2 are 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, and 24.

The first twelve multiples of 4 are 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, and 48.

The first twelve multiples of 6 are 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, and 72.

Of the first twelve multiples of 2, 4 and 6, the common multiples are 12 and 24. The least common multiple is 12.

Example  What is the LCM of 3 and 5?

Multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, 27, 30 . . . .

Multiples of 5 are 5, 10, 15, 20, 25, 30 . . . .

The smallest common multiple is 15; thus, the LCM is 15.

The LCM of several numbers may be determined by obtaining the prime factorizations of each of the numbers. The LCM consists of the product of the highest power of each factor present. This procedure is given here.

<table>
<thead>
<tr>
<th>HOW TO USE PRIME FACTORIZATIONS TO FIND THE LCM OF NUMBERS GIVEN</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1.</strong> Write the prime factorizations of all the numbers. The repeated factors should be represented with exponents.</td>
</tr>
<tr>
<td><strong>2.</strong> Identify the highest power of each factor present.</td>
</tr>
<tr>
<td><strong>3.</strong> The LCM is equal to the product of the highest powers of each factor present.</td>
</tr>
</tbody>
</table>
Example
Find the LCM of 100 and 120.

The prime factorization of 100 is $2^2 \times 5^2$.

The prime factorization of 120 is $2^3 \times 3 \times 5$.

The factors present are 2, 3, and 5. The highest powers of the factors present are $2^3$, $3^1$, and $5^2$. The LCM consists of the product $2^3 \times 3 \times 5^2$ which equals $8 \times 3 \times 25 = 600$.

Example
Find the LCM of 24, 39, and 45

The prime factorization of 24 is $2^3 \times 3$.

The prime factorization of 39 is $3 \times 13$.

The prime factorization of 45 is $3^2 \times 5$.

The factors present are 2, 3, 5, and 13. The highest powers of the factors present are $2^1$, $3^2$, 5, and 13. The LCM consists of the product $2^1 \times 3^2 \times 5 \times 13$, which is equal to $8 \times 9 \times 5 \times 13 = 4680$.

Another method for finding the LCM of several numbers consists of taking multiples of the largest number given and testing to see if the multiple is divisible by all of the numbers given. The method is described here.

HOW TO FIND THE LCM OF NUMBERS GIVEN BY USING MULTIPLES

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<table>
<thead>
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<tbody>
<tr>
<td>1.</td>
<td>Take multiples of the largest number, beginning with 1 times the number. Test each multiple to see if it is evenly divisible by all of the numbers given.</td>
</tr>
<tr>
<td>2.</td>
<td>The LCM consists of the smallest multiple that is evenly divisible by all of the numbers given.</td>
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<tr>
<td></td>
<td>Note: In some cases, it is much easier to use the prime factorization method.</td>
</tr>
</tbody>
</table>
**Example** Find the LCM of 100 and 120 by using multiples.

Since 120 is the larger number, we take multiples of 120 and check each multiple to see if it is divisible by 100.

\[
120 \times 1 = 120 \quad \text{120 is not divisible by 100.}
\]
\[
120 \times 2 = 240 \quad \text{240 is not divisible by 100.}
\]
\[
120 \times 3 = 360 \quad \text{360 is not divisible by 100.}
\]
\[
120 \times 4 = 480 \quad \text{480 is not divisible by 100.}
\]
\[
120 \times 5 = 600 \quad \text{600 is divisible by 100. 600 is the LCM.}
\]

**Example** Find the LCM of 210 and 110 by using multiples.

The multiples of 210, the larger number, are 210, 420, 630, 840, 1050, 1260, 1470, 1680, 1890, 2100, and 2310.

The multiple 2310 is divisible by 110. Thus, 2310 is the LCM.

Note that in this example it would have been easier to use the prime factorizations of 210 and 110 to find the LCM.

**Example** Find the LCM of the algebraic quantities \(A \times B\) and \(B \times C\) where \(A\), \(B\), and \(C\) represent real numbers. These quantities are already in prime factored form. Use the method of prime factorizations. (*This is how LCM is used in Algebra!*)

Since these quantities are already written in prime factored form, the prime factorizations are \(A^2 \times B\) and \(B^2 \times C\).

The factors present are \(A\), \(B\), and \(C\).

The highest powers of the factors present are \(A^2\), \(B^2\), and \(C\).

The LCM of \(A^2 \times B\) and \(B^2 \times C\) is the product \(A^2 \times B^2 \times C\).

**MATH FACT**

- The method of using prime factorizations to find the LCM must be used in algebraic fraction problems.
Using the Least Common Multiple in Fraction Addition and Subtraction

When adding or subtracting fractions, all of the fractions must have the same denominator. Thus, the first step in fraction addition and subtraction consists of determining what the common denominator should be. It is desirable to obtain a common denominator that is as small as possible. Since we will build each fraction into a fraction with the common denominator, the common denominator must be a multiple of each of the denominators. This means that the least common denominator must be equal to the LCM of all of the denominators. The least common denominator is abbreviated LCD.

<table>
<thead>
<tr>
<th>MATH FACT</th>
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<tbody>
<tr>
<td>The least common denominator is equal to the LCM of all of the denominators.</td>
</tr>
</tbody>
</table>

**Example** Find the least common denominator (LCD) of \( \frac{3}{14} \) and \( \frac{5}{17} \). Use prime factorizations to find the LCM of the denominators.

The LCD is equal to the LCM of 14 and 17.

The prime factorization of 14 is \( 2 \times 7 \).
The prime factorization of 17 is \( 17 \times 1 \).

The factors present are 2, 7, and 17. The LCM is \( 2 \times 7 \times 17 = 238 \). The LCD is also 238.

**Example** Find the least common denominator (LCD) of \( \frac{3}{50}, \frac{13}{40}, \) and \( \frac{7}{100} \). Take multiples of the largest denominator to find the LCM of the denominators.

The LCD is equal to the LCM of 50, 40 and 100.

The multiples of 100 are 100, 200, 300, etc. The smallest multiple of 100 that is divisible by both 50 and 40 is 200.

The LCD of these fractions is 200.
Example

Add \( \frac{3}{14} + \frac{5}{88} \) by first finding the LCD. Build each fraction into an equivalent fraction with the common denominator. Then, combine the fractions. Find the LCM of the denominator by using the prime factorization method.

The LCD is equal to the LCM of 14 and 88.

The LCM of 14 and 88 is obtained by using the prime factorizations of 14 and 88.

The prime factorization of 14 is \( 2 \times 7 \).

The prime factorization of 88 is \( 2^3 \times 11 \).

The LCM of 14 and 88 is \( 2^3 \times 7 \times 11 = 8 \times 7 \times 11 = 616 \).

\[
\frac{3}{14} = \frac{3 \times 44}{14 \times 44} = \frac{132}{616} \quad \text{Multiply by 44 since } 616 \div 14 = 44.
\]

\[
\frac{5}{88} = \frac{5 \times 7}{88 \times 7} = \frac{35}{616} \quad \text{Multiply by 7 since } 616 \div 88 = 7.
\]

\[
\frac{132}{616} + \frac{35}{616} = \frac{132 + 35}{616} = \frac{167}{616}.
\]

In this example, the LCM of 14 and 88 could have also been determined by finding the smallest multiple of 88 that is divisible by both 14 and 88. \( 7 \times 88 = 616 \) and 616 is divisible by 14. The least common multiple is 616.

Example

Subtract \( \frac{31}{100} - \frac{11}{150} \) by first finding the LCD. Build each fraction into an equivalent fraction with the common denominator and then subtract the fractions. Use the multiples method to find the LCM.

The LCD is equal to the LCM of 100 and 150.

The LCM of 100 and 150 may be obtained by finding the smallest multiple of 150 that is divisible by 100. Multiples of 150 are 150, 300, 450, 600, etc. The smallest multiple that is divisible by 100 is 300.

300 is the LCM of 100 and 150. Also, 300 is the LCD.

Continued on next page . . .
\[
\frac{31}{100} = \frac{31 \times 3}{100 \times 3} = \frac{93}{300}
\]
Multiply by 3 since \(300 \div 100 = 3\).

\[
\frac{11}{150} = \frac{11 \times 2}{150 \times 2} = \frac{22}{300}
\]
Multiply by 2 since \(300 \div 150 = 2\).

\[
\frac{93}{300} - \frac{22}{300} = \frac{93 - 22}{300} = \frac{71}{300}.
\]

The following are algebraic fraction problems. You are not responsible for these, but these examples illustrate why we learn the LCM method!

**Example**

Find the LCD of the algebraic fractions \(\frac{1}{B^2 \times C}\) and \(\frac{1}{C^2 \times D}\) where \(B, C,\) and \(D\) represent real numbers. The denominators are already in prime factored form.

The LCD of the denominators is equal to the LCM of \(B^2 \times C\) and \(C^2 \times D\).

The denominators are already written in prime factored form. Thus, the prime factorizations are \(B^2 \times C\) and \(C^2 \times D\).

The factors present are \(B, C,\) and \(D\).

The highest powers of these factors are \(B^2, C^2,\) and \(D\).

The LCM of \(B^2 \times C\) and \(C^2 \times D\) is the product \(B^2 \times C^2 \times D\).

The LCD is \(B^2 \times C^2 \times D\).

**Example**

Add the algebraic fractions \(\frac{1}{A^2 \times B}\) and \(\frac{1}{B^3 \times D}\) where \(A, B,\) and \(D\) represent real numbers. The denominators are already in prime factored form.

The LCD is equal to the LCM of \(A^2 \times B\) and \(B^3 \times D\).

The denominators are already written in prime factored form. Thus, the prime factorizations are \(A^2 \times B\) and \(B^3 \times D\).

The factors present are \(A, B,\) and \(D\).

The highest powers of these factors are \(A^2, B^2,\) and \(D\).
The LCM of $A^2 \times B$ and $B^2 \times D$ is the product $A^2 \times B^2 \times D$.

The LCD is $A^2 \times B^2 \times D$.

Now, write each fraction as an equivalent fraction with the LCD of $A^2 \times B^2 \times D$.

The numerator and the denominator of $\frac{1}{A^2 \times B}$ are multiplied by $B \times D$ because $(A^2 \times B) \times (B \times D) = A^2 \times B^2 \times D$.

$$\frac{1 \times B \times D}{A^2 \times B} \times \frac{B \times D}{B \times D} = \frac{B \times D}{A^2 \times B^2 \times D}$$

The numerator and the denominator of $\frac{1}{B^2 \times D}$ are multiplied by $A^2$ because $(B^2 \times D) \times A^2 = B^2 \times D \times A^2 = A^2 \times B^2 \times D$.

$$\frac{1 \times A^2}{B^2 \times D} \times \frac{A^2}{A^2} = \frac{A^2}{A^2 \times B^2 \times D}$$

The two fractions can be added by combining the numerators.

$$\frac{B \times D}{A^2 \times B^2 \times D} + \frac{A^2}{A^2 \times B^2 \times D} = \frac{B \times D + A^2}{A^2 \times B^2 \times D}$$

$\frac{B \times D + A^2}{A^2 \times B^2 \times D}$ is the final answer.

**Comparing Fractions**

If you were asked the question, “Which is more, $\frac{3}{4}$ of a gallon or $\frac{14}{20}$ of a gallon?”, you could answer this question by writing both fractions with a common denominator.

The least common denominator of these two fractions is 20. The fraction $\frac{3}{4}$ may be written as an equivalent fraction with a denominator of 20 by multiplying the numerator and the denominator by 5.

$$\frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20}.$$

Thus, $\frac{3}{4} = \frac{15}{20}$ is greater in value than $\frac{14}{20}$.

To compare fractions, use the following procedure.
**PROCEDURE TO COMPARE FRACTIONS**

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<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>1.</td>
<td>Write all of the fractions with a common denominator.</td>
</tr>
<tr>
<td>2.</td>
<td>Compare the numerators of the fractions. The fraction with the larger numerator is greater in value.</td>
</tr>
</tbody>
</table>

**Example**

List the fractions $\frac{7}{10}$, $\frac{11}{16}$, and $\frac{13}{40}$ in order from smallest to largest.

First, it is necessary to find a common denominator. Since $40 \times 2 = 80$ is divisible by all of the denominators, 80 is the LCD.

All of the fractions must be rewritten as equivalent fractions with a denominator of 80.

$$
\frac{7}{10} = \frac{7 \times 8}{10 \times 8} = \frac{56}{80}
$$

$$
\frac{11}{16} = \frac{11 \times 5}{16 \times 5} = \frac{55}{80}
$$

$$
\frac{13}{40} = \frac{13 \times 2}{40 \times 2} = \frac{26}{80}
$$

Now, compare numerators. Since 26 is the smallest numerator, $\frac{26}{80}$ is the smallest fraction. The next smallest fraction is $\frac{55}{80}$ and the fraction of largest value is $\frac{56}{80}$. Thus, if we list the original fractions in order from smallest to largest we obtain $\frac{13}{40}$, $\frac{11}{16}$, $\frac{7}{10}$. 
Methods of Adding and Subtracting Fractions

In the previous section, the procedure for adding or subtracting fractions required that all fractions have the same denominator. The procedure is given again here.

**PROCEDURE TO ADD OR SUBTRACT FRACTIONS**

1. If the fractions have different denominators, find a common denominator, and build new equivalent fractions with the common denominator.

2. Combine the numerators of the new fractions with the addition or subtraction operations present. Write this result over the common denominator.

3. Reduce the fraction result to lowest terms if possible.

Many students find it difficult to calculate a common denominator. The least common denominator is the best common denominator, and it is necessary that the LCD be used for algebraic fractions. For numerical fractions, *any* common denominator will suffice for fraction addition and subtraction.

For example, \( \frac{1}{12} + \frac{3}{14} \) may be added by using the least common denominator of 84. Also, \( \frac{1}{12} + \frac{3}{14} \) may be added by using the common denominator of 168.

Using the LCD of 84, \( \frac{1}{12} + \frac{3}{14} = \frac{1 \times 7}{12 \times 7} + \frac{3 \times 6}{14 \times 6} = \frac{7}{84} + \frac{18}{84} = \frac{25}{84} \).

Using the common denominator of 168, \( \frac{1}{12} + \frac{3}{14} = \frac{1 \times 14}{12 \times 14} + \frac{3 \times 12}{14 \times 12} = \frac{14}{168} + \frac{36}{168} = \frac{50}{168} \) which reduces to \( \frac{25}{84} \).

As you can see from this example, the same result is obtained whether the least common denominator or any common denominator is used. There are different methods of finding a common denominator. These methods are summarized on the following page.
### METHODS OF FINDING A COMMON DENOMINATOR

- **Write the prime factorization of each denominator.** The LCD is equal to the product of the highest power of each factor present.
  
  This method *must* be used when adding or subtracting algebraic fractions.

- **Take multiples of the largest denominator.** The LCD is equal to the smallest multiple that is divisible by all of the denominators.

- **Multiply all of the denominators together.** This product will always be a common denominator, however, it may not be the LCD.

---

**Example**

Find a common denominator of the fractions, $\frac{1}{4}$, $\frac{3}{5}$, and $\frac{7}{10}$.

The LCD of 20 is quickly obtained by taking multiples of the largest denominator, 10. The product of all of the denominators, 200, is also a common denominator, but not the LCD.

**Example**

Subtract the fractions $\frac{10}{21} - \frac{7}{20}$. Use all three methods to find a common denominator.

The LCD of these fractions may be obtained by taking multiples of 21.

These multiples are 21, 42, 63, 84, 105, 126, 147, 168, 189, 210, 231, 252, 273, 294, 315, 336, 357, 378, 399, and 420.

This is not a very efficient method since the only multiple of 21 is divisible by 20 is $20 \times 21 = 420$.

The LCD may be determined by using prime factorizations of the denominators.

$21 = 3 \times 7$

$20 = 2^2 \times 5$

The LCD is equal to $2^2 \times 3 \times 5 \times 7 = 420$.

*Continued on next page.*
A common denominator may also be determined by multiplying the denominators together. The product $20 \times 21$ produces a common denominator of 420. In this case, the product of the denominators resulted in the LCD because the denominators shared no common factors.

The fractions must now be written as equivalent fractions with denominators of 420. Then, the fractions can be combined with addition.

\[
\begin{align*}
\frac{10}{21} &= \frac{10 \times 20}{21 \times 20} = \frac{200}{420} \\
\frac{7}{20} &= \frac{7 \times 21}{20 \times 21} = \frac{147}{420}
\end{align*}
\]

\[
\frac{200}{420} - \frac{147}{420} = \frac{200 - 147}{420} = \frac{53}{420}
\]

**MATH FACT**

- If there is no factor common to any two of the denominators, then the LCD will be equal to the product of the denominators.

To see why this “math fact” is true, note that when no two denominators share a common factor, no two of the prime factorizations of the denominators will contain a common factor. Thus, the LCD will consist of all the factors from all of the denominators. For example, if the denominators of two fractions are 15 and 49, the prime factorizations are $3 \times 5$ and $7^2$. Note that 15 and 49 do not share a common factor and the $\text{LCD} = 3 \times 5 \times 7^2 = 15 \times 49 = 735$.

**Example**

Add the fractions $\frac{1}{12} + \frac{1}{7} + \frac{3}{11}$.

No two of the denominators 12, 7, and 11 share a common factor. In other words, no two of these denominators are divisible by the same counting number. The LCD will be $12 \times 7 \times 11 = 924$.

\[
\begin{align*}
\frac{1}{12} &= \frac{1 \times 77}{12 \times 77} = \frac{77}{924} \\
\frac{1}{7} &= \frac{1 \times 132}{7 \times 132} = \frac{132}{924} \\
\frac{3}{11} &= \frac{3 \times 84}{11 \times 84} = \frac{252}{924}
\end{align*}
\]

\[
\frac{77}{924} + \frac{132}{924} + \frac{252}{924} = \frac{77 + 132 + 252}{924} = \frac{461}{924}
\]
Here is another algebra problem - again, you are not responsible for this in this class but you may see this type of problem in future classes.

**Example**

Add the algebraic fractions \( \frac{1}{A^2 \times B^3} + \frac{1}{A^5 \times B^2} \). Note that the denominators are already in prime factored form.

Prime factorizations *must* be used to find the LCD of these fractions.

The prime factorizations of the denominators are \( A^2 \times B^3 \) and \( A^5 \times B^2 \).

The largest powers of the factors present are \( A^5 \) and \( B^3 \). The LCD is equal to the product \( A^5 \times B^3 \).

The numerator and the denominator of \( \frac{1}{A^2 \times B^3} \) must be multiplied by \( A^3 \) in order to convert this fraction into an equivalent fraction with a denominator of \( A^5 \times B^3 \).

\[
\frac{1}{A^2 \times B^3} \times \frac{(A^3)}{(A^3)} = \frac{A^3}{A^2 \times A^3 \times B^3} = \frac{A^3}{A^5 \times B^3}
\]

Note that the *commutative property of multiplication* allowed us to rewrite the denominator of \( A^2 \times B^3 \times (A^3) \) as \( A^2 \times A^3 \times B^3 \).

The numerator and the denominator of \( \frac{1}{A^5 \times B^2} \) must be multiplied by \( B \) in order to convert this fraction into an equivalent fraction with a denominator of \( A^5 \times B^3 \).

\[
\frac{1}{A^5 \times B^2} \times \frac{(B)}{(B)} = \frac{B}{A^5 \times B^3}
\]

The final answer is \( \frac{A^3}{A^5 \times B^3} + \frac{B}{A^5 \times B^3} = \frac{A^3 + B}{A^5 \times B^3} \).
Addition and Subtraction of Mixed Numbers

Mixed numbers may be added or subtracted by converting them into improper fractions and then adding or subtracting the fractions by using previously learned methods. Another method of adding or subtracting mixed numbers consists of adding or subtracting the whole number parts, adding or subtracting the fractional parts, and then combining the results. These two methods are summarized here with examples following each method.

**METHOD 1 - ADDING OR SUBTRACTING MIXED NUMBERS**

1. Convert all mixed numbers into improper fractions.
2. Add or subtract the improper fractions using fraction addition and subtraction methods.

**Example**

Add $\frac{3}{8} + \frac{5}{6}$.

\[
\frac{3}{8} = \frac{3 \times 8 + 1}{8} = \frac{25}{8}
\]
\[
\frac{5}{6} = \frac{4 \times 6 + 5}{6} = \frac{29}{6}
\]

These two fractions can be added after each is written as an equivalent fraction with a common denominator of 24.

\[
\frac{25}{8} = \frac{25 \times 3}{8 \times 3} = \frac{75}{24}
\]
\[
\frac{29}{6} = \frac{29 \times 4}{6 \times 4} = \frac{116}{24}
\]

\[
\frac{75}{24} + \frac{116}{24} = \frac{191}{24} \text{ or } 7\frac{23}{24}
\]

**Example**

Subtract $\frac{35}{5} - \frac{11}{40}$.

\[
\frac{35}{5} = \frac{35 \times 5 + 2}{5} = \frac{177}{5}
\]
\[
\frac{11}{40} = \frac{11 \times 40 + 7}{40} = \frac{447}{40}
\]
\[
\frac{177}{5} = \frac{177 \times 8}{5 \times 8} = \frac{1416}{40}
\]
\[
\frac{1416}{40} - \frac{447}{40} = \frac{969}{40} \text{ or } 24\frac{9}{40}
\]
The second method for adding or subtracting mixed numbers consists of adding and subtracting the whole number and fraction parts. This method is given here.

**METHOD 2 - ADDING OR SUBTRACTING MIXED NUMBERS**

1. **Add or subtract the fraction parts of the mixed numbers.**
   
   **Note:** If the fraction being subtracted is too large, borrow a fractional equivalent of 1 from the first whole number and add it to the first fraction.

2. **Add or subtract the whole number parts of the mixed numbers.**

3. **Combine the results of steps 1 and 2 as a mixed number.**

**Example**

Add \(3 \frac{1}{8} + 4 \frac{5}{6}\). (This is the same problem given in Example 1.)

Add the fraction parts of the mixed numbers.

\[
\frac{1}{8} + \frac{5}{6} = \frac{1 \times 3}{8 \times 3} + \frac{5 \times 4}{6 \times 4} = \frac{3}{24} + \frac{20}{24} = \frac{23}{24}
\]

Add the whole number parts of the mixed numbers.

\(3 + 4 = 7\)

Now, combine the two results.

\(7 + \frac{23}{24} = 7 \frac{23}{24}\)

**Example**

Add \(2 \frac{1}{5} + 5 \frac{3}{4}\).

Add the fraction parts of the mixed numbers.

\[
\frac{1}{5} + \frac{3}{4} = \frac{4}{20} + \frac{15}{20} = \frac{19}{20}
\]

Add the whole number parts of the mixed numbers.

\(2 + 5 = 7\)

Combine the results.

\(7 + \frac{19}{20} = 7 \frac{19}{20}\)

**Example**

Subtract \(24 \frac{3}{5} - 11 \frac{7}{8}\).
Subtract the fraction parts of the mixed numbers.

\[
\frac{3}{5} - \frac{7}{8} = \frac{24}{40} - \frac{35}{40} = ?
\]

Since \( \frac{35}{40} \), the fraction being subtracted, is larger than \( \frac{24}{40} \), we must add a fractional equivalent of 1 which is \( \frac{40}{40} \) to \( \frac{24}{40} \). The amount, 1, is “borrowed” from 24, the whole number of the first mixed number. This process is shown here.

\[
24 \frac{23}{40} - 11 = 13 \frac{12}{40}
\]

\[
\left( \frac{24}{40} + \frac{40}{40} \right) - \frac{35}{40} = \frac{64}{40} - \frac{35}{40} = \frac{29}{40}
\]

The final answer is \( 12 \frac{29}{40} \).

**Example**

Add \( \frac{7}{8} + 3 \frac{6}{7} \).

\[
\frac{7}{8} + \frac{6}{7} = \frac{49}{56} + \frac{48}{56} = \frac{97}{56}
\]

\( 2 + 3 = 5 \)

Adding 5 to \( \frac{97}{56} \) results in \( 5 \frac{97}{56} \). In general, a mixed number is never written with an improper fraction. This answer should be written as

\[
5 \frac{97}{56} = 5 + \frac{41}{56} = 6 \frac{41}{56}.
\]